Random square-tiled surfaces of large genus and random multicurves on surfaces of large genus

with V. Delecroix, P. Zograf, A. Zorich

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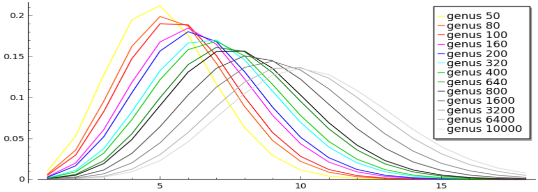
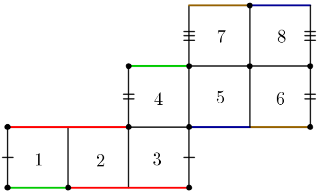


Summary of the previous talk

Square-tiled surfaces



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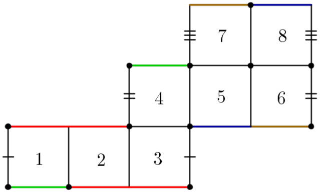
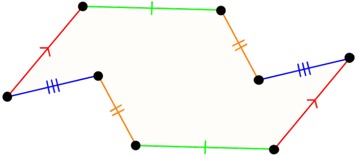


as a special case of translation surfaces



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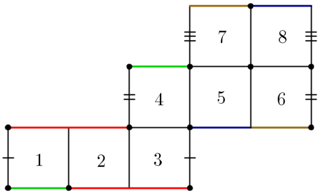

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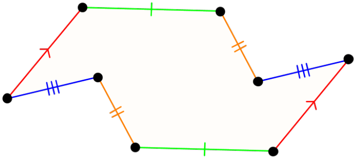

Asymptotically as the number of squares grows, square-tiled surfaces with fixed combinatorics equidistribute in the moduli space, and the horizontal and vertical combinatorics become uncorrelated.

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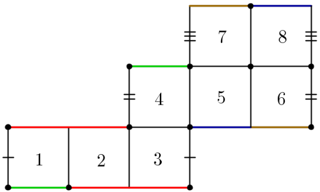

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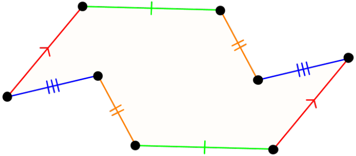
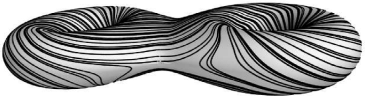
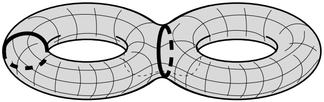


Multicurves on hyperbolic surfaces

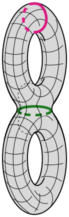


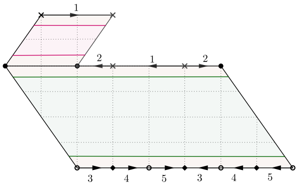


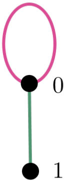
Multicurves VS Square-tiled surfaces

Frequencies of SQT VS multicurves on surfaces

For a square-tiled surface, the core curves of the horizontal cylinders form a reduced multicurve on the surface.







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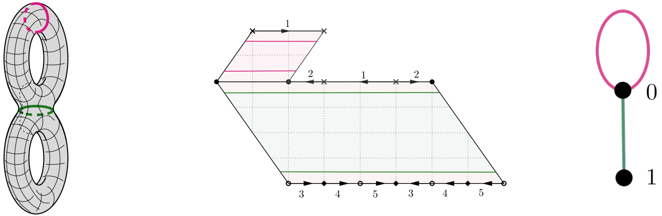
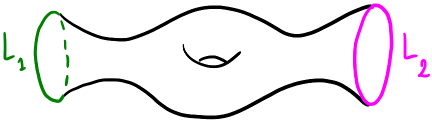


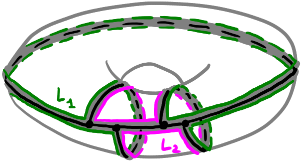
Fact: The frequency $c(\gamma_0)/b_g$ of multicurves of type γ_0 and the frequency c/Vol of SQTs of corresponding topological type **coincide**!

Examples: 1-component multicurves/ 1-cylinder SQTs, Separating curves/separating cylinders, etc.

Why frequencies are the same?

Hyperbolic surface with boundaries VS Ribbon graph





$\mathrm{Vol}_{WP}\,\mathcal{M}_{g,n}(L_1,\ldots,L_n)$ VS $N_{g,n}(L_1,\ldots,L_n)$

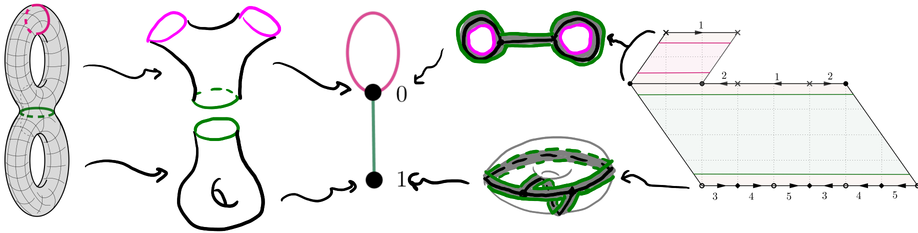
(Mirzakhani) (Kontsevich)

$$\sim \sum_{\alpha\vdash 3g-3+n} \frac{\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1}\ldots\psi_n^{\alpha_n}}{\alpha_1!\ldots\alpha_n!} L_1^{2\alpha_1}\ldots L_n^{2\alpha_n} \text{ as } L_i\to\infty$$

Why are frequencies the same?

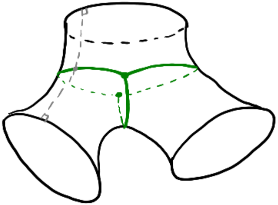
Cut hyperbolic surfaces along geodesics:

Cut (half-translation) SQTs along cylinders:



The pieces are glued together along the same ”stable” graph (topological type of the multicurve / the decomposition into cylinders).

Why are frequencies the same?



$\mathcal{M}_{g,n}(\mathbf{L})$ moduli space of genus g hyperbolic surfaces with geodesic boundaries of length $\mathbf{L} = (L_1, \ldots, L_n)$
$\mathcal{M}^*_{g,n}(\mathbf{L})$ moduli space of genus g ribbon graphs with face lengths $\mathbf{L}$

- [Bowditch-Epstein '88] The spine map $\mathcal{S}$ is a homeomorphism between $\mathcal{M}_{g,n}(\mathbf{L})$ and $\mathcal{M}^*_{g,n}(\mathbf{L})$
- [Do '10] In the Gromov–Hausdorff topology, $\forall \Gamma \in \mathcal{M}^*_{g,n}(\mathbf{L})$,

$$\lim_{N\to\infty} \frac{1}{N}\mathcal{S}^{-1}(N\Gamma) = \Gamma.$$

- [Mondello '09, Do '10] The pullback of the normalized Weil-Petersson form $\frac{\omega}{N^2}$ on $\mathcal{M}_{g,n}(N\mathbf{L})$ by $f : \Gamma \mapsto \mathcal{S}^{-1}(N\Gamma)$ converges pointwise to the Kontsevich 2-form on $\mathcal{M}^*_{g,n}(\mathbf{L})$.

Intersection numbers on $\mathcal{M}_{g,n}$

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$\overline{\mathcal{M}_{g,n}}$ Deligne-Mumford compactification
Holomorphic line bundle L_i (fibers=cotangent complex lines to X at P_i)
First Chern class $\psi_i = c_1(L_i)$
For $d_1 + \cdots + d_n = 3g - 3 + n$ define

$$\langle \tau_{\mathbf{d}} \rangle = \langle \tau_{d_1} \dots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}_{g,n}}} \psi_1^{d_1} \dots \psi_n^{d_n}$$

Witten's conjecture: they satisfy certain recurrence relations which are equivalent to certain differential equations on the associated generating function ("partition function in 2-dimensional quantum gravity "). Proved by M. Kontsevich; alternative proofs belong to A. Okounkov and R. Pandharipande, to M. Mirzakhani, to M. Kazarian and S. Lando (and there are more).

Aggarwal's proof of large genus asymptotics of intersection numbers

Theorem (Aggarwal '20)

As $g \to \infty$ and $n = o(g^{1/2})$,

$$\langle \tau_{\boldsymbol{d}} \rangle_{g,n} = \int_{\overline{\mathcal{M}_{g,n}}} \prod_i \psi_i^{d_i} \simeq \frac{(2|\boldsymbol{d}|+1)!!}{24^g g! \prod_i (2d_i+1)!!}$$

Explicit formula for $n = 1$ [Kontsevich].
Virasoro constraints

$$\langle \tau_{\mathbf{d}} \rangle_{g,n+1} = \sum_i A_i \langle \tau_{\mathbf{d}^{(i)}} \rangle_{g,n} + \sum_j B_j \langle \tau_{\mathbf{d}^{(j)}} \rangle_{g-1,n+2} + \sum_{k,l} C_{k,l} \langle \tau_{\mathbf{d}^{(k)}} \rangle_{g',n'+1} \langle \tau_{\mathbf{d}^{(l)}} \rangle_{g-g',n-n'+1}.$$

Interlude on random integers and random permutations

Number of prime divisors of random integers

Theorem (Prime Number Theorem)

An integer number n taken randomly in a large interval $[1, N]$ *is prime with asymptotic probability* $\frac{\log N}{N}$*.*

Denote by $\omega(n)$ the number of prime divisors of an integer n counted without multiplicities, i.e., for $n = p_1^{m_1} \dots p_k^{m_k}$, $\omega(n) = k$.

Theorem (Erdös–Kac CLT)

$$\lim_{N \to \infty} \frac{1}{N} \operatorname{card} \left\{ n \leq N, \ \frac{\omega(n) - \log \log N}{\sqrt{\log \log}} \leq x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt.$$

(rate of convergence described by A. Rényi and P. Turán ('58), and of A. Selberg ('54))

Number of cycles of a random permutation

$\mathrm{K}_n(\sigma)$: number of cycles in the cycle decomposition of $\sigma \in S_n$.

- $\mathbb{P}\big(\mathrm{K}_n(\sigma) = k\big) = \frac{s(n,k)}{n!}$, where $s(n,k)$ is the unsigned Stirling number of the first kind. In particular $\mathbb{P}\big(\mathrm{K}_n(\sigma) = 1\big) = \frac{1}{n}$.
- [Goncharov, '44] As $n \to +\infty$:

$$\mathbb{E}(\mathrm{K}_n(\sigma)) = \log n + \gamma + o(1)\,, \qquad \mathbb{V}(\mathrm{K}_n(\sigma)) = \log n + \gamma - \zeta(2) + o(1)\,,$$

and CLT:

$$\lim_{n\to+\infty} \frac{1}{n!}\,\mathrm{card}\left\{\sigma \in S_n \,\middle|\, \frac{\mathrm{K}_n(\sigma) - \log n}{\sqrt{\log n}} \leq x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt\,.$$

Number of cycles of a random permutation

For a random variable X taking values in $\mathbb{Z}_+$,

$$\mathbb{E}(t^X) = \sum_{k=1}^{\infty} \mathbb{P}(X = k) t^k.$$

Example : Poisson distribution of parameter λ

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad \mathbb{E}(t^X) = e^{\lambda(t-1)}$$

For X and Y independent, $\mathbb{E}(t^{X+Y}) = \mathbb{E}(t^X)\mathbb{E}(t^Y)$.

Definition

X_n converges mod-Poisson with parameters λ_n and limiting function $G(t)$ if $\exists R > 1$, $\varepsilon_n \to 0$, $\forall t \in \mathbb{C}$ such that $|t| < R$,

$$\mathbb{E}(t^{X_n}) = e^{\lambda_n(t-1)} G(t) \big(1 + O(\varepsilon_n)\big)$$

Number of cycles of a random permutation

Theorem (Hwang '94, Nikeghbali-Zeindler '13)

$\mathrm{K}_n(\sigma)$ *converge mod-Poisson with parameters* $\lambda_n = \log(n)$ *and limiting function* $G(t) = \frac{t}{\Gamma(1+t)}$*, that is:*
For any $t \in \mathbb{C}$*, as* $n \to \infty$ *we have*

$$\mathbb{E}(t^{\mathrm{K}_n(\sigma)}) = e^{\log(n)\cdot(t-1)} \cdot \frac{t}{\Gamma(1+t)} \cdot \left(1 + O\left(\frac{1}{n}\right)\right).$$

Consequences for $\mathrm{K}_n(\sigma)$:

1. asymptotic expansion of moments,
2. central limit theorem,
3. local limit theorem,
4. large deviations.

Shape of a random multicurve

Random multicurves and square-tiled surfaces

Fixing a genus g, choosing the uniform measure on all integral multicurves of length at most L, and letting L tend to infinity we define a "random multicurve" on a surface of genus g, via Mirzakhani's result:

$$s_X(L,\Gamma) \sim B(X) \cdot \frac{c(\gamma)}{b_g} \cdot L^{6g-6}, \text{ where } b_g = \sum_{[\gamma]} c(\gamma).$$

In this setting we interpret $\frac{c(\gamma)}{b_g}$ as the probability for a random multicurve to have type γ.

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In this setting we interpret $\frac{c(\gamma)}{b_g}$ as the probability for a random multicurve to have type γ.

- Does a random multicurve separates the surface ?
- What is the number of primitive components of a random multicurve ?
- Is a random multicurve primitive ?

Random multicurves and square-tiled surfaces

In the same way fixing a genus g, choosing the uniform measure on all square-tiled surfaces with at most N squares and letting N tend to infinity we define a "random square-tiled surface via the following asymptotics:

$$\mathrm{card}\{SQT \text{ with } \leq N \text{ squares of type } \Gamma\} \sim c(\Gamma)N^{6g-6}.$$

Here $\frac{c(\Gamma)}{\mathrm{Vol}(\mathcal{Q}_g)}$ is the probability for a random square-tiled surface to have type Γ.

Results: Non-separateness and primitivity

Theorem (Delecroix-G-Zograf-Zorich)

Consider a random multicurve $\gamma = \sum_{i=1}^{k} m_i \gamma_i$ on a surface S of genus g. Let $\gamma_{red} = \gamma_1 + \cdots + \gamma_k$ be the underlying reduced multicurve. The following asymptotic properties hold as $g \to +\infty$.

(a) *The probability that γ_{red} does not separate the surface (i.e. $S - \sqcup \gamma_i$ is connected) tends to 1.*

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(b') *For any positive integer m, the probability that all weights m_i of a random multicurve $\gamma = m_1\gamma_1 + m_2\gamma_2 + \ldots$ on a surface of genus g are bounded by a positive integer m (i.e. that $m_1 \le m, m_2 \le m, \ldots$) tends to $\sqrt{\frac{m}{m+1}}$ as $g \to +\infty$.*

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(c) *For any sequence of positive integers k_g with $k_g = o(\log g)$ the probability that a random multicurve $\gamma = \sum_{i=1}^{k_g} m_i \gamma_i$ is primitive (i.e. that $m_1 = \cdots = m_{k_g} = 1$) tends to 1 as $g \to +\infty$.*

Results : distribution of the number of components K_g

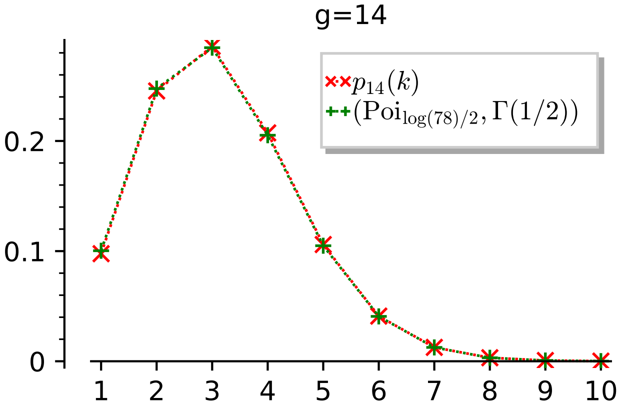
Theorem (Delecroix-G-Zograf-Zorich)

$\mathrm{K}_g(\gamma)$ converge mod-Poisson with parameters $\lambda_g = \frac{\log(6g-6)}{2}$ and limiting function $G(t) = \frac{t \cdot \Gamma(\frac{3}{2})}{\Gamma(1+\frac{t}{2})}$ as $g \to \infty$, that is:
For all $t \in \mathbb{C}$ such that $|t| < \frac{8}{7}$ the following asymptotic relation is valid as $g \to +\infty$:

$$\mathbb{E}\left(t^{\mathrm{K}_g(\gamma)}\right) := \sum_{k=1}^{3g-3} \mathbb{P}(\mathrm{K}_g(\gamma) = k) t^k = e^{\lambda_g(t-1)} \cdot \frac{t \cdot \Gamma(\frac{3}{2})}{\Gamma(1+\frac{t}{2})} \left(1 + o(1)\right),$$

where $\lambda_g = \frac{\log(6g-6)}{2}$. Moreover, for any compact set U in the open disc $|t| < \frac{8}{7}$ there exists $\delta(U) > 0$, such that for all $t \in U$ the error term has the form $O(g^{-\delta(U)})$.

Results: Mod-Poisson convergence



Exact distribution of number of components (coeffs of $\mathbb{E}(t^{\mathrm{K}_g(\gamma)})$)
Mod-Poisson convergence (coeffs of $e^{\lambda_g(t-1)} \cdot G(t)$)

Consequence 1: CLT for number of components K_g

Theorem (Delecroix-G-Zograf-Zorich)

Choose a non-separating simple closed curve ρ_g on a surface of genus g. Denote by $\iota(\rho_g, \gamma)$ the geometric intersection number of ρ_g and γ. The centered and rescaled distribution defined by the counting function $\mathrm{K}_g(\gamma)$ tends to the normal distribution:

$$\lim_{g\to+\infty} \sqrt{\frac{3\pi g}{2}} \cdot 12g \cdot (4g-4)! \cdot \left(\frac{9}{8}\right)^{2g-2}$$
$$\lim_{N\to+\infty} \frac{1}{N^{6g-6}} \operatorname{card} \left(\left\{ \gamma \in \mathcal{ML}_g(\mathbb{Z}) \,\middle|\, \iota(\rho_g, \gamma) \leq N \quad \textit{and} \right.\right.$$
$$\left.\left. \frac{\mathrm{K}_g(\gamma) - \frac{\log g}{2}}{\sqrt{\frac{\log g}{2}}} \leq x \right\} / \operatorname{Stab}(\rho_g) \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt\,.$$

Consequence 2: local limit theorem

Theorem (Delecroix-G-Zograf-Zorich)

Let $\lambda_g = \log(6g-6)/2$*. For any* $x \in [0, 1.23)$ *we have uniformly in* $0 \leq k \leq x\lambda$

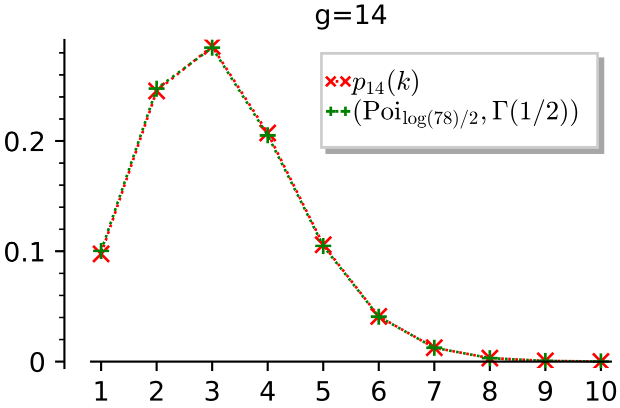
$$\mathbb{P}(\mathrm{K}_g(\gamma) = k+1) = \frac{e^{-\lambda_g}\lambda_g^k}{k!} \cdot \left(G\left(\frac{k}{\lambda_g}\right) + O\left(\frac{k}{\lambda_g^2}\right)\right).$$

+ Explicit formula for the tail $\mathbb{P}(\mathrm{K}_g(\gamma) > x\lambda_g + 1)$

Expansion of the moments, in particular:

$$\mathbb{E}(\mathrm{K}_g(\gamma)) = \lambda_g + \frac{\gamma}{2} + \log 2 + o(1),$$
$$\mathbb{V}(\mathrm{K}_g(\gamma)) = \lambda_g + \frac{\gamma}{2} + \log 2 - \frac{3}{4}\zeta(2) + o(1),$$

Consequence 2: local limit theorem

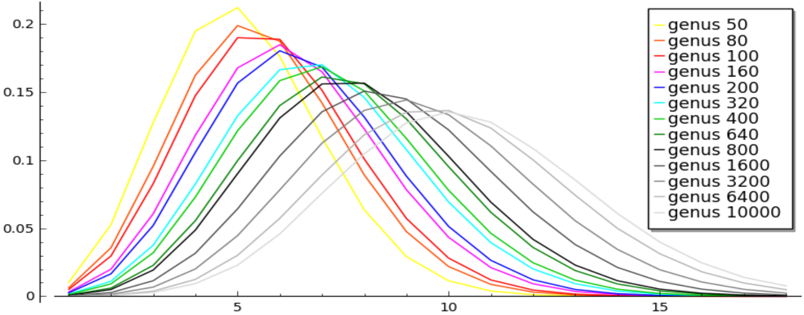


Exact distribution of number of components: $p_g(k) = \mathbb{P}(\mathrm{K}_g(\gamma) = k)$

Local limit theorem: $\frac{e^{-\lambda_g} \lambda_g^k}{k!} G\left(\frac{k}{\lambda_g}\right)$

Comparison with random permutations

	Number of cycles of random permutations [Goncharov], [Hwang] $\mu_n = \log n$ $\tilde{G}(t) = \frac{t}{\Gamma(1+t)}$	Number of components of random multicurves $\lambda_g = \frac{\log(6g-6)}{2}$ $\tilde{\gamma} = \frac{\gamma}{2} + \log(2)$ $G(t) = \frac{t\cdot\Gamma(\frac{3}{2})}{\Gamma(1+\frac{t}{2})}$
$\mathbb{E}(K)$	$\mu_n + \gamma + o(1)$	$\lambda_g + \tilde{\gamma} + o(1)$
$\mathbb{V}(K)$	$\mu_n + \gamma - \zeta(2) + o(1)$	$\lambda_g + \tilde{\gamma} - \frac{3}{4}\zeta(2) + o(1)$
CLT	ok	ok
$\mathbb{E}(t^K)$	$e^{\mu_n(t-1)}\tilde{G}(t)\,(1 + o(1))$	$e^{\lambda_g(t-1)}G(t)\,(1 + o(1))$
$p_K(k+1)$	$\frac{e^{-\mu_n}\mu_n^k}{k!}\left(\tilde{G}(\frac{k}{\mu_n}) + O\left(\frac{k}{\mu_n^2}\right)\right)$	$\frac{e^{-\lambda_g}\lambda_g^k}{k!}\left(G(\frac{k}{\lambda_g}) + O\left(\frac{k}{\lambda_g^2}\right)\right)$


0.2
0.15
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genus 100
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genus 320
genus 400
genus 640
genus 800
genus 1600
genus 3200
genus 6400
genus 10000